Teaching locus with a conserved property by integrating mathematical tools and dynamic geometric software

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We present investigative tasks that concern loci, which integrate the use of dynamic geometry software (DGS) with mathematics for proving the obtained figures. Additional conditions were added to the loci: ellipse, parabola and circle, which result in the emergence of new loci, similar in form to the original loci. The mathematical relation between the parameters of the original and new loci was found by the learners. A mathematical explanation for the general case, using the 'surprising' results obtained in the investigative tasks, is presented. Integrating DGS in mathematics instruction fosters an improved teaching and learning process.

Introduction

A conserving property may be used to solve different problems in mathematics and the identification of such a property leads to a complete and deep understanding of mathematical concepts and the relationships between them. The locus, according to this definition, conserves a certain property and can therefore be used to prove a wide variety of problems and even serve in the execution of construction problems using a straightedge and a compass, as has been done in mathematics since antiquity.

Loci can be divided into two groups:

1. Essential loci, such as: the straight line, parallel lines, the mid perpendicular of a segment, an angle bisector, the circle, the parabola, the ellipse and the hyperbola.
2. Loci which are a result of the essential loci or loci obtained as a result of satisfying several mathematical conditions.

Analytical geometry is one prominent field in mathematics in which extensive use is made of loci.

In most colleges for teacher training, prospective teachers of the high school (ages 12–17 years old) stream in mathematics are taught four sequential courses, which deal with loci.

The first course is the elements of Euclidean geometry. The second course is the principles of analytical geometry which begins with the straight line and ends with the circle. The third course is integration of computer technology in the teaching of mathematics where the students acquire the skills of using different computerised technological tools. The fourth course is advanced analytical geometry, which begins with the parabola, the ellipse and the hyperbola and ends with unique subjects, and deals with loci, making maximum use of the knowledge acquired in the three previous courses, and in particular, the use of computer technology. Advanced analytic geometry is intended to demonstrate the importance of the locus as an important tool in the mathematical toolbox of solving different problems where identification of conserved properties allows one to obtain a mathematical generalisation. In this subject the loci are a basis for additional tasks which include more requirements, and allow one to obtain new loci based on the given loci.

Sometimes an introduction of additional requirements brings about formation of a locus whose shape is the same as that of the original locus, and other times a different locus is formed. The present paper deals with obtaining loci which are similar in shape to the original locus.

In some cases mathematical conditions are given and the student is required to discover what essential locus is obtained.

Naturally in all cases the students were required to prepare formal mathematical proofs for the locus shape obtained. It should be emphasised that the students must know that what is seen on the computer screen is not mathematical proof, but only a basis for speculation.

In some tasks the students found several proofs as a result of using different mathematical tools, such as: Euclidean geometry, analytical geometry, trigonometry and even differential calculus. Finding different solutions for the same problem accentuated the interconnectedness of mathematics for the students.

Development of pedagogical content knowledge

Pedagogical content knowledge (PCK) has become one of the popular constructs studied by many scholars since it was defined by Shulman (1986). This knowledge includes the knowledge of what is acceptable to teach as a repertoire of modes of representation, the most effective forms of representation, powerful analogies, metaphors, explanations and demonstrations, which makes learning easy or difficult, knowledge (research and experimentation) on the perceptions and misconceptions of concepts, teaching strategies for coping and effective methods of reorganisation of knowledge.

Ball and Bass (2003), relied on various types of knowledge identified by Shulman and defined the concept of mathematical knowledge for teaching (MKT), as knowledge supporting the ideas linked by the teacher and

emphasising integrating, the ability to plan and evaluate mathematical tasks, together with the mathematical knowledge required to integrate and manage these tasks.

Pedagogical knowledge of teachers has a significant role in leading and implementing reforms in mathematics education. Teachers ask for the use of innovative technology to bring together students, working with tasks and meaningful activities, creating a community in which students coping with a given problem can discuss, explain, connect and reflect on their learning process (NCTM, 1991, 2000).

Pedagogical knowledge of teachers includes selection, construction and adaptation of mathematical tasks in a classroom environment. The process of construction and integration of tasks through teaching and learning develops existing knowledge, and incorporates mathematics and pedagogy for the participants. Teachers engaged in solving tasks are also invited to reflect on existing knowledge and their interaction with the task (Zaslavsky, 2007).

Stein, Smith and Henningsen (2000) described mathematical task classroom activity, which focuses student's attention on a particular mathematical idea. Classroom activity involves problems related to the same mathematical idea

Tasks that call for complex learning processes, called powerful tasks, are necessary for the development of knowledge and the teaching of teachers (Kranier, 1993).

Powerful tasks are meaningful rather than routine tasks that lead to collaboration and social interaction, and reveal mathematical thinking and pedagogical links. They are challenging, and encourage independent thought, and belief in processes of understanding of mathematical concepts. One model of powerful tasks is combining technological tools and conceptual understanding (Zaslavsky, Chapman & Leikin 2003).

Use of computer technology

Research in education seeks ways to improve the quality of teaching and learning, and it is therefore also focused on the integration of technology in teaching. Technological tools allow one to construct and represent mathematical objects in a dynamic manner while providing the user with feedback during the solution of problems (Alakoc, 2003; Martinovic & Manizade, 2013). Learning that includes use of dynamic software allows the students to discover mathematical phenomena, as well as mathematical models. In addition, it encourages different representations and connections between graphical descriptions while referring to mathematical concepts (Wiest, 2001). Learning by using a technological tool allows students to better describe mathematical concepts and relations when compared to teaching that does not include a technological tool. In one study, for example, students showed greater understanding of mathematical concepts and were

given access to high-level mathematical ideas (Hohenwarter, Hoherwarter & Lavicza, 2008).

The investigative task presented in the present paper has been given to different groups of students as part of the course Advanced Analytical Geometry. The aim of this task is to teach students to understand deeply and eventually teach the subject of loci. Also it models the integration of technology in teaching the subject. The students were given several tasks and were asked to investigate them both from the regular mathematical aspect and through the use of dynamic geometry software, which in the present study was *Geogebra*.

The use of the dynamic software allows the students to solve problems by learning from examples. The students draw conclusions from examples concerning the essential moves, track the critical properties of the concept, internalise it and implement it subsequently in the solution of new problems. The computer's ability to generate multiple diversified examples rapidly, to store and retrieve moves and to provide quality feedback, provides the learner with information on the mathematical concept that constitutes the basis for the generalisations and hypotheses that require proof (Chazan, 1993; Dreyfus & Hadas, 1996; Hanna, 2000; Laborde, 2000).

The use of computer technology allowed students to make dynamic changes in the data and obtain the locus by dragging. It also enhanced students' ability to realise the change immediately.

Teachers who integrate technology through teaching must be flexible during the discussions. They also need to be attentive to speculations that relate inherently to the subject taught during the lesson in order to give students the feeling that their way of thinking is a vital contribution to a broader view of the mathematical world. Once the student finds a hypothesis, or a phenomenon, there is more process-based understanding (Lampert, 1993).

Stage A of the investigative task

The students were required to construct three applets for loci using dynamic software (*Geogebra*), which are based on essential loci whose parameters are given by particular numeric values.

1. Construct a canonic ellipse which intersects the x-axis and the points A and B. Select some point C on the ellipse, so that it can be dragged along the curve. A triangle CAB is obtained. Connect the point C with the point O, the origin. The segment OC is a median in the triangle. Place by construction the point M, the point of intersection of the medians in the triangle CAB, on the segment OC, so that by moving the point C the point M shall move correspondingly. The question is: what is the locus on which M moves? Correct construction of the applet shows that the point M moves on a canonic ellipse as shown in Figure 1.

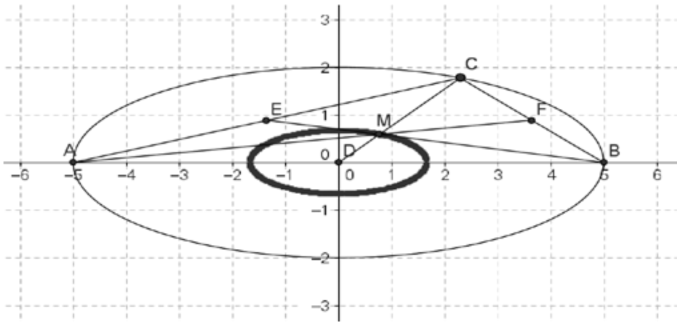

Figure 1

2. Construct a parabola whose vertex lies on the y-axis which intersects the x-axis at the points A and B. Select any point C on the parabola so that it can be dragged along the curve. A triangle CAB is obtained. The point C is connected with point O, the origin, and the segment OC is a median in the triangle. Place by construction the point M, the point of intersection of the medians in the triangle CAB, on the segment OC, so that by moving the point C, the point M shall move accordingly. The question is: what is the locus on which M moves? Correct construction of the applet shows that the point M moves on a parabola whose vertex lies on the y-axis, as shown in Figure 2.

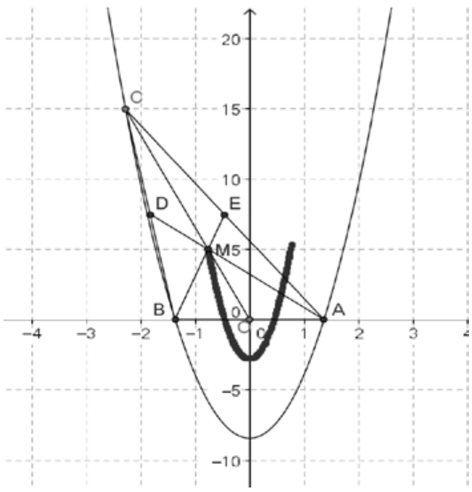

Figure 2

3. Construct the circle based on the coordinates of its centre and its radius. Select a fixed point A on the circle, and construct from it at least 6–8 arbitrary chords, followed by a construction of the midpoint M of each chord: M_1, M_2, M_3… The question is: what is the locus on which point M moves?
Correct construction of the applet shows that point M moves on a circle, as shown in Figure 3.

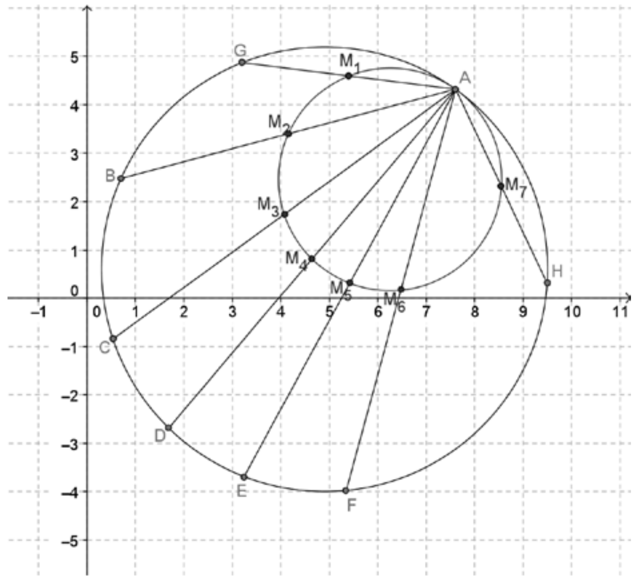

Figure 3

The resulting conclusion is that in each of the cases the point *M* moves on a locus that is similar to the original locus.

Stage B of the advanced investigative task

The students were asked to investigate tasks 1–2 for the case in which point *M* divides the segment *OC* in a ratio other than 1:2 (intersection of medians), and task 3 with point *M* dividing the chords in a ratio different than 1:1 (middle of the chord).

First surprising result

Upon completion of the correct construction of the applet the students found an interesting phenomenon: for each ratio of segment division it became clear that point *M* continues to move on a locus that is similar to the original locus.

Applets for tasks

Available online are *Geogebra* applets in which the parameters of the loci and the division ratio of the segments can be changed:

- Locus in an ellipse: https://www.geogebratube.org/student/m149059
- Locus in a parabola: http://tube.geogebra.org/student/m149060
- Locus in a circle: http://tube.geogebra.org/student/m149061

In each of the applets there are rulers that allow one to change the parameters of the locus and the division ratio.

The students were required to present proofs for each of the above cases. Following are mathematical proofs for the cases of an ellipse and a parabola:

Mathematical proof for the formation of an ellipse inside an ellipse

Given the canonic ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

as shown in Figure 1, points A and B mark the points of intersection of the ellipse with the x-axis. Point C (x, y) moves on the ellipse and point M moves correspondingly on the segment OC, so that in any case the following ratio is maintained between the segments:

$$\frac{MC}{OM}=\frac{n}{m}=k$$

The coordinates of point M are (X, Y), as shown in Figure 4.

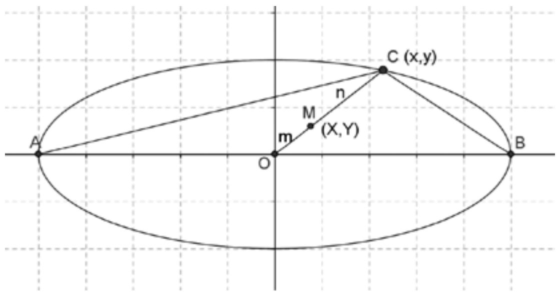

Figure 4

The coordinates of point M in accordance with the subdivision of a segment in a given ratio are:

$$X=\frac{mx}{m+n},\ Y=\frac{my}{m+n}$$

Hence we obtain that: $x = (k + 1)X$, $y = (k + 1)Y$. When we substitute these values in the equation for the ellipse, we obtain:

$$\frac{X^2}{\left(\frac{a}{k+1}\right)^2}+\frac{Y^2}{\left(\frac{b}{k+1}\right)^2}=1$$

In other words, point M moves on the perimeter of a canonic ellipse, the lengths of whose axes is smaller by a factor of $(k + 1)$ than the lengths of the axes of the original ellipse.

Mathematical proof for the formation of a parabola inside a parabola

Given is the parabola $y = x^2 - a^2$ that intersects the x-axis at the points $A(a, 0)$ and $B(-a, 0)$.

We select on the parabola a point $C(x, y)$ and connect it to the points A and B, so that the triangle ABC is obtained. CO (where O is the origin) is a median in the triangle ABC. We mark on the median a point M so that

$$\frac{MC}{OM} = \frac{n}{m} = k$$

We move point C on the parabola, and ask on which geometrical shape will point M move?

Based on the coordinates of the subdivision of the segment, we obtain:

$$X = \frac{mx}{m+n}, Y = \frac{my}{m+n}$$

Hence we obtain that $x = (k + 1)X$, $y = (k + 1)Y$. By substituting the values of x in the equation of the parabola we obtain a new equation of a parabola

$$Y = (k+1)X^2 - \frac{a^2}{k+1}$$

This parabola intersects the x-axis at the points

$$\left(\frac{a}{k+1}, 0\right) \text{ and } \left(-\frac{a}{k+1}, 0\right)$$

Thus, point M moves on a locus which is a parabola (Figure 5). The formed parabola is narrower, but its vertex is also located on the y-axis.

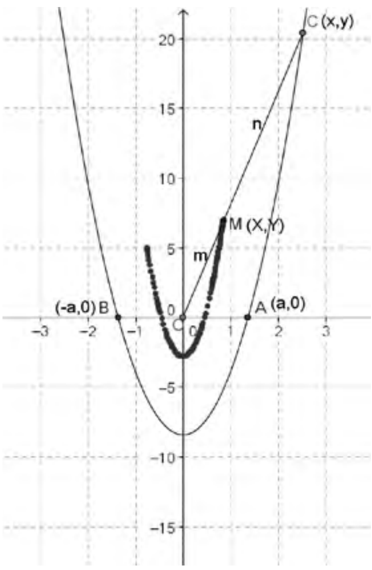

Figure 5

The proofs demonstrate a connection between the parameters of the obtained locus and the parameters of the original locus as a function of the division ratio.

Stage C of the investigative task

During this stage the students were asked whether the point from which the segments connecting to the points on the original locus can be anywhere inside the locus, such that point M will still move on a similar locus.

Most of the students thought that the answer was no and some said that they did not know.

A very surprising result

An arbitrary point P is selected inside the locus, from which straight lines are drawn to the points A_1, A_2, A_3… on the original locus, and subsequently the division points M_1, M_2, M_3… are marked with a constant ratio on the formed segments. It turns out that these points are located on a locus with the same shape as the original locus.

An example of this case is shown in Figure 6, where point P is located at some location inside the circle and points M move along the perimeter of the internal circle.

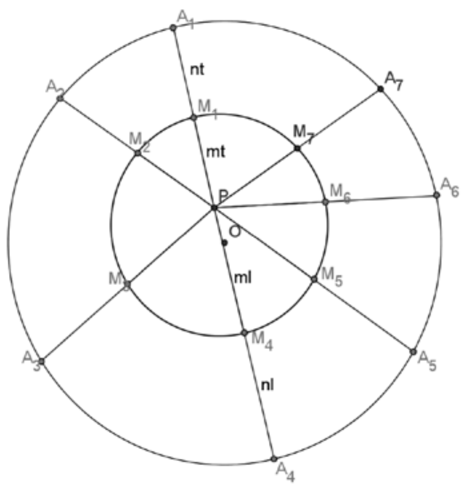

Figure 6

There is a mathematical relation between the radius of the original circle in the radius of the obtained circle as a function of the division ratio k:

$$\frac{n}{m} = k$$

Point O is the centre of the circle.

$$t(m + n) + l(m + n) = 2R \qquad (1)$$

$$(t + l)(m + n) = 2R \qquad (2)$$

$$m(t + l) = 2r \qquad (3)$$

By dividing expression (3) by expression (2), we obtain:

$$r = \frac{m}{n+m}R \text{ or } r = \frac{1}{k+1}R$$

An applet for the formation of a circle inside a circle is available at http://tube.geogebra.org/student/m149062. In the applet one can move point P and change the division ratio k using the ruler.

The general case: An arbitrary locus

Point C moves on a given curve. We select some point O outside the curve and connect it by straight lines with the points C_1, C_2, C_3… on the curve.

On the segments OC_1, OC_2, OC_3… we mark the points M_1, M_2, M_3… that divide each segment in the same ratio $\frac{m}{n}$, i.e.:

$$\frac{OM_1}{M_1C_1} = \frac{OM_2}{M_2C_2} = \frac{OM_3}{M_3C_3} = \ldots = \frac{m}{n}$$

On what curve do the points M_1, M_2, M_3… move?

We consider the general case. A point O is given in the plane. In addition a real number $k \neq 0$ is given. Let us consider the transformation of the plane which transfers each point C to a point C', so that

$$\frac{OC'}{OC} = k \left(\text{or } \frac{OC'}{C'C} = \frac{k}{k-1} \right)$$

This transformation is called a homothety with a centre O and a factor k. The homothety transfers some curve (shape) G into a curve G', so that G' is similar to G.

In other words the locus of all the points C' that lie on the segments connecting the point O with the points C that lie on the curve, such that

$$\frac{OC'}{OC} = k$$

is a curve G' that is similar to G (Figure 7).

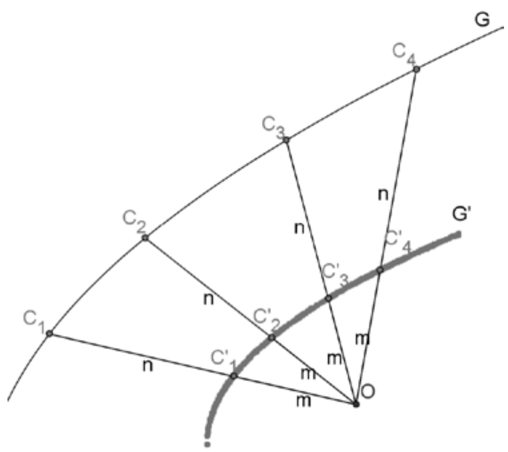

Figure 7

Hence it is clear that if:

a) point C moves on a given curve,

b) we select some arbitrary point O connected by straight lines, with the points C_1, C_2, C_3… on the curve,

c) we mark points C'_1, C'_2, C'_3… on the segments OC_1, OC_2, OC_3… which divide each segment in the same ratio $\frac{m}{n}$.

In other words,

$$\frac{OC'_1}{C'_1C_1} = \frac{OC'_2}{C'_2C_2} = \frac{OC'_3}{C'_3C_3} = \ldots = \frac{m}{n}$$

then the points C_1, C_2, C_3… move on a curve that is similar to the given curve.

The importance of the transformation

According to the definition of the great mathematician Felix Klein, geometry is "a science that investigates the properties of shapes that do not change under transformations of the plane (space)" (Klein, 2015).

The use of different transformations, whether basic or complex, allows one in many cases to obtain a simpler solution of problems in plane or in analytic geometry.

In the studies of plane geometry in high school (ages 12–17) use is made of auxiliary constructions and even simple alternative constructions, but almost no use is made of transformations of shift, reflection relative to a straight line or a point, rotation or homothety.

The solution of even simple problems using transformations develops geometric insight, provides alternative tools and illustrates how transformation of a geometrical shape can be carried out.

Making the transformation strengthens the notion that a geometrical shape is not a 'frozen' object, but rather something that can be moved.

The special approach to the solution of tasks having to do with loci is intended to provide and impart additional tools allowing one to deal with mathematical challenges, sometimes in a simple manner when compared to commonly used methods.

Different transformations are known. One transformation preserves distances: it is called an isometric transformation. When compared to such transformations, homothety is a transformation relative to a centre O, which matches each point X with a point X' on the straight line OX, and therefore $OX' = k \cdot OX$, where k is called the similarity factor.

If k is positive, the points X and X' are located on the same side of the centre O, and if k is negative, the two points are located on both sides of the centre O. Homothety is also a basic transformation but it is not isometric because it does not preserve a constant distance between the points unless $k = 1$.

The potential of combining mathematical tools and computerised technological tools is very powerful since it allows dynamic learning accompanied by visual representation of the different parameters affecting the shapes of the loci.

We explain the phenomena mentioned throughout this article using homothety.

In all examples cited in the article transformation was applied on all the locus points.

The image of each point A on the geometric location is a point A', which maintains $OA' = k \times OA$ when O is a given point and k is a positive real number. It seems that this transformation, which is called homothety, contains two features that explain the phenomena discovered in the investigation:

1. If A' and B' are respectively images of points A and B, then there is $A'B' = k \times AB$. In other words, the section $A'B'$ is the shrinking (or expansion) of AB with respect to k.
   This attribute stems from imaginary triangles $DOA'B' \sim DOAB$ (Figure 8).

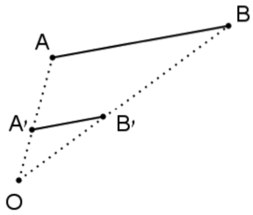

Figure 8

2. If A, B and C are on one line, then the pictures obtained through transformation can also be found on one line. This attribute also stems from the similarity of triangles: Because $\alpha = \alpha'$ and $\beta = \beta'$, if $\alpha + \beta = 180°$ then $\alpha' + \beta' = 180°$ (Figure 9).

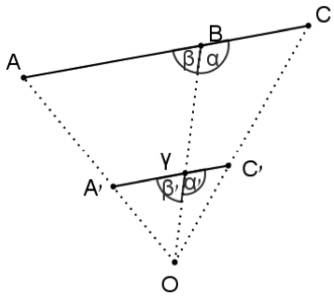

Figure 9

These features of transformation characterise photography magnification and thus explain the similarity between the original curve and the curve obtained through transformation.

Another layer of explanation is possible if we recall the definitions of loci.

A circle is the locus of all points with equal distances from a given point. The centre of the circle is a given size. Think of a circle whose centre P and the radius R. If P is the centre of the circle, obtained through transformation, then the curve obtained by dragging gathers all the dots that are equidistant from P' constant and equal to $k \times R$, so we got a new circle with radius $k \times R$ (Figure 10).

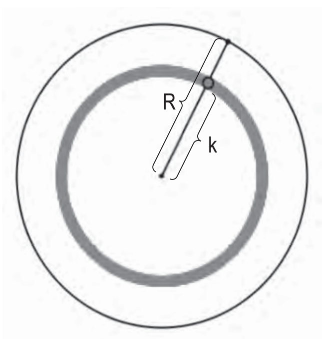

Figure 10

Similarly, an ellipse is the locus of all the points where the sum of their distances from two given points is fixed. They will be P' and Q' images of the focal ellipse. If the M point on the original ellipse and M' image obtained from using transformation will result that $M'P' + M'Q' = k \times MP + k \times MQ = k(MP + MQ)$.

In other words, all the points on the curve obtained by using the drag engage the defining feature of the ellipse: the sum of distances of the points P' and Q' is a fixed size, $k(MP + MQ)$.

Similarly, we can show that in the case of the parabola, the curve obtained by dragging reserves the geometric features of the place (Figure 11).

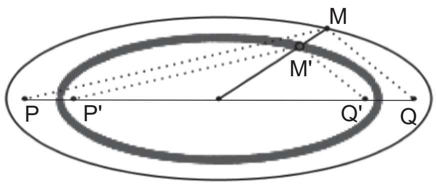

Figure 11

In fact, homothety is a transformation preserving similarity of any graph and any geometric shape, as we can see in the following examples:

Examples of homothety using dynamic software

Example A: The graph of an algebraic function

We take the graph of an algebraic function, for example $f(x) = x^3 - 9x^2$. We pick some point A outside the graph and connect it with some point B on the graph of the function. We divide the segment AB by some ratio

$$k\frac{BP}{BA} = k$$

By dragging point B along the graph of the function and tracing point B, we obtain a curve that is similar to the original curve, as seen in Figure 12.

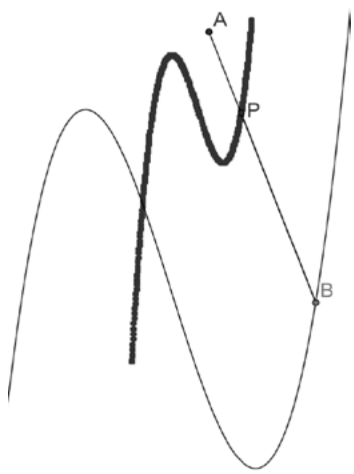

Figure 12

The creation of the similar locus on the graph of an algebraic function is available at http://tube.geogebra.org/student/m149066.

This demonstrates the creation of the similar locus on the graph of an algebraic function. The applet contains rulers that allow changing the value of the parameter of the function and the division factor.

Example B: Demonstrating the property on an arbitrary triangle
Given is an arbitrary triangle ABC. We select some point P inside the triangle. We connect this point by straight lines to points on the sides of the triangle. On each of the obtained segments we construct a corresponding point M which divides the segment by the ratio k. The points M obtained using this method are located on the sides of a triangle that is similar to the original triangle, as seen in Figure 13.

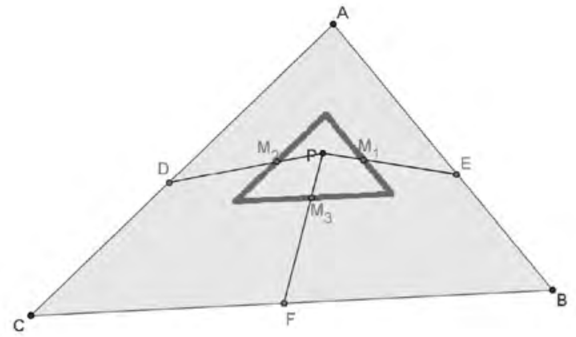

Figure 13

The applet for the formation of a triangle inside a triangle is available at http://tube.geogebra.org/student/m149069.

This applet demonstrates the method of obtaining a similar triangle when the location of the vertices of the triangle and the location of the point P can be changed. There is also a ruler that allows the division ratio to be changed. By dragging the points D, E, F along the sides AC, AB, BC respectively and tracing the points M_1, M_2, M_3, the sides of the inner triangle that is similar to the original triangle ABC are formed.

The students were required to check the existence of these properties in other geometrical shapes as well.

Similar parabolas and constant segment ratio

Given are two parabolas with a common vertex at the origin $y = ax^2$ and $y = bx^2$ with a, $b > 0$ and $a > b$, as shown in Figure 14. The straight line $y = cx$ that passes through the origin intersects the parabolas at the points A and B.

Prove that the ratio of the lengths of the segments $\frac{BA}{AO}$ is constant and does not depend on c.

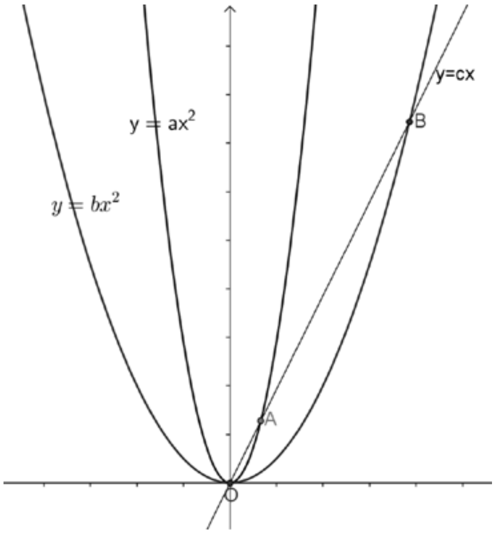

Figure 14

By equating the equation of the straight line and the equations of the parabolas, we obtain the points of intersection

$$A\left(\frac{c}{a},\frac{c^2}{a}\right),\ B\left(\frac{c}{b},\frac{c^2}{b}\right)$$

Hence we obtain

$$BA=\frac{c}{\frac{1}{b}-\frac{1}{a}}\cdot\sqrt{1+c^2},\ AO=\frac{c}{a}\cdot\sqrt{1+c^2}$$

and therefore

$$\frac{BA}{AO}=\frac{a}{b}-1$$

In other words, the ratio $\frac{BA}{AO}$ remains constant and does not depend on the slope of the straight line. Thus, any parabola $y = ax^2$ $(a > 0)$ is similar to the parabola $y = bx^2$ $(b > 0)$, and the two parabolas maintain a constant ratio between the lengths of the chords formed by straight lines issuing from the origin.

In the same manner, in the case that one of the parabolas has a minimum $y = ax^2$ $(a > 0)$ and the other one has a maximum $y = bx^2$ $(y = bx^2)$, and both share a common vertex, as shown in Figure 15, the property of the constant ratio between the lengths of the segments $\frac{BA}{AO}$ is maintained. The calculation of the lengths of the segments gives the ratio

$$\frac{BA}{AO}=\frac{a}{b}+1$$

It should be noted that the action of creating the similar parabola is natural, since when taking a parabola of the type $y = ax^2$, and multiplying it by a real number $k \neq 0$, we obtain a similar parabola which is narrower or wider or inverted, depending on the value and the sign of k. What is surprising is that the ratio of the segments between two similar parabolas is constant and is a conserved property.

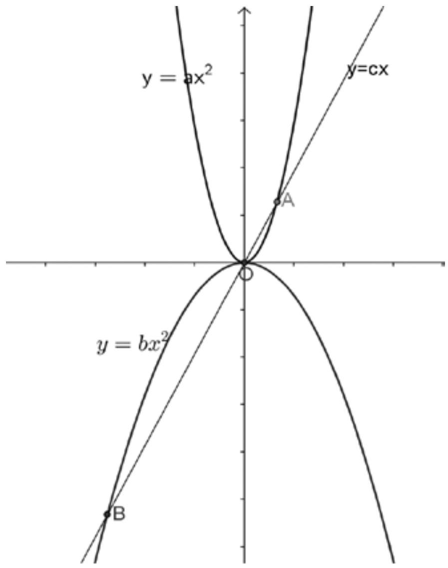

Figure 15

The methodical aspect

The paper presents investigative activity that deals with finding loci that conserve the shape of the original locus, where usually in the class environment the students deal with tasks having to do with finding loci without considering possible generalisations. Several stages of investigative activity have been presented. During the first stage use was made of dynamic geometric software through which the students could see the actual creation of the locus in a dynamic manner, while dragging a point on the original locus and tracing the point that is located at the original locus. (The 'tracing' operation is one of the possibilities provided for by the dynamic software (*Geogebra*) in order to allow tracking the movement of a point while dragging a different point). Work with dynamic software allowed the students to gain a better and more profound understanding of the concept 'locus', and especially of the dynamics of its construction, the process that the students and the pupils of the class find hard to see when the task is given statically in a textbook. The algebraic solution also presents algebraic processes that do not sufficiently aid dynamic comprehension of the locus. In the second stage, as in the subsequent stages, the use of the dynamic software allowed the students to identify surprising generalisations that demonstrate the conservation of the original locus, even when the division ratio of the relevant segment changes and the location of the original point through which the pencil of lines passes changes. Martinovic and Manizade (2013) describe the contribution of dynamic software as part of the recurring pedagogic process: reflexive pedagogy in action. Of the different roles of dynamic software, they choose to accentuate the role of dynamic software as a partner in the learning process. The use of the dynamic software accelerates processes of understanding and mathematical justification. In this case users rely on their own mathematical knowledge in order to justify what the technology exemplifies.

Stupel & Ben-Chaim (2013) suggested the concept "semi-proof" in working with a technological tool while dragging different points and objects, identifying a conserving property, generalising, etc. The dragging operation offers a dynamic view of mathematical objects; identification of properties and relations between them which remain constant while creating a collection of multiple special case examples and hypothesising (De Villiers, 1998). Teaching in a technological environment poses a pedagogic challenge in the presentation of the importance and necessity of mathematical proof to the process of generalisation. In other words: in making it clear to students that a semi-proof based on identification of phenomena and generalisations while working with that the ecological tool is not enough, and stressing the importance of valid mathematical proof to the infinite range of special cases.

The investigative task presented in the paper is a powerful task that exposes the students to a complex process of learning and to associating mathematical and pedagogic reasoning (Krainer, 1993). The processes of construction of the objects using the dynamic software and the processes of mathematical generalisation put an emphasis on the link between different subjects in mathematics: Euclidean geometry (segment ratios, triangle similarity) and analytic geometry (locus, division of the segment at a given ratio). Exposing the students to relations of these types provides them with an opportunity to consider mathematics as an interconnected science and not as a discrete collection of disjoint subjects (House & Coxford, 1995; NCTM, 2000).

In addition, the investigative activity presented in the paper offers students the opportunity to perform generalisation of mathematical phenomena, from which they can draw conclusions with regard to similar tasks.

As mathematics teacher educators who are charged with the professional development of teachers, we have presented pre-service teachers and two groups of practising teachers with an activity that is suitable for two teaching environments: the teaching environment of teachers (academic educational college) and the teaching environment of pupils in class. In order to improve teaching, the students and the teachers were presented with an investigative activity that is relevant to the class environment. The way the students and the teachers dealt with the task and the discussions held with regards to mathematical and pedagogic aspects of the task provide the opportunity for expanding pedagogic and mathematical knowledge (Segal, Stupel & Oxman, 2015). Tasks that are relevant for integration in the class environment have led the teachers to find more interest, to cooperate, to be surprised, to reflect on the methods of their teaching in the class environment and to interact with the tasks. In order to achieve these goals the tasks for the teachers must be similar to the tasks given in the classroom environment (Zaslavsky, 2007, 2008).

The selected loci examples demonstrated in this article, are to our knowledge original and are very important since they represent the beauty of mathematics, which has essential value in mathematical education.

Participants' responses in their own words

Investigative activity has been presented before 45 students as part of two study groups in a course of advanced analytic geometry, and to 20 practicing teachers as part of teacher training. In addition, 20 mathematics teacher-educators were presented with this activity as part of national conferences. All the participants were surprised by the mathematical discoveries to which they were exposed. They experienced in practice the strength of the technological tool as a significant partner in the investigative activity. The ability to construct with relative ease corresponding applets that describe the similar curves furthered the awareness on several levels of all those e solving the investigative task in combination with the technological tool:

- The students' level: the technological tool as an assistant in significant and deeper understanding of complex and dynamic mathematical concepts and phenomena.
- The teachers' level: the importance of using the technological tool for integrating investigative activity in teaching in the classroom environment.
- The level of teachers' teachers: the importance of exposing teachers' teachers to investigative activity that combines the computerised technological tool for the purpose of instructing future teachers.

Following are some examples of comments given by the different populations, which reflected their awareness of the contribution of the computerised tool:

The comments of students

Evidence of the processes specified above were obtained while conducting a reflective discussion with the students concerning the processes they experienced while solving the activity in the technological environment. Some comments of the students follow:

- "Now (with integration of the dynamic software) I see and understand what a locus means, how a locus is formed. It was difficult to imagine before."
- "The computerised tool presents in a visual and clear manner the property of the new locus which is similar to the original locus, which is very surprising."
- "Working with the computer allowed us to change the values of the parameters and to see their influence on the locus immediately., We could then subsequently try to understand why this is always true."
- "Working with dynamic software allowed us to discover surprising properties that can be generalised."

The comments of teachers

- "I will do my best to integrate the technological tool in my lessons because of the immediate visual impact, the surprising nature and the aesthetics of the task, which contribute to the understanding of the study material."

- “Despite all the contribution of the technological computerised tool, due to the workload in the program of studies, I do not have the time for the students to acquire the skills for using the tool. At most I will use it as a presentation to demonstrate to the students without them experiencing it independently.”
- “The school must make sure that the computerised technological tools are accessible in each class and offer the students the option to use them freely; thus we will bring them closer to the digital age even in the classroom.”
- “The task gave me a new view of the teaching of loci.”

The words of teacher educators

- “The effectiveness and the importance of the computerised tool proved itself in many diverse areas, and this is the right time to integrate it in the educational system as a powerful tool.”
- “The use of technology tools must be included in most courses taught to pre-service teachers in mathematics education, because it improves the quality of teaching and brings about significant learning.”
- “The task allowed us to encounter generalisation processes which are usually hard to recognise, but are important for improving the quality of teaching.”

Summary

We presented investigative tasks of creating a new locus through the addition of data that relate to the original locus. In each task, when DGS software was used, a surprising discovered feature was that the new locus is similar to the original locus and there is a connection between the parameters. The activity resulted in conserving properties that can be generalised. Full mathematical proofs were given that corroborate the results of the surprising graphical presentation, and point to the importance of the computer tool.

Mathematics teachers require a broad PCT and MKT training which includes the use of different representations of mathematical objects, dynamic use of technology suitable for integrating tasks that combine exploration processes and mathematical proofs. The tasks presented in this paper are powerful tasks, which demand the discovery of surprising conserved properties of the curve or a geometric shape.

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